Abstract

The Internship last for 8 weeks, during this time one project named 2D partial slip contact takes two weeks and was finished. The other one is still carried on and become my fourth year project. This report will briefly introduce what are those projects and what did I learned from them.

1 Partial Slip Contact problem

1.1 Introduction

Imagine you have a cylinder and you press it to a rigid surface (i.e. a block), you will find that the contact surface area will increase with the force of your press. The study of such behaviour is called the contact problem, which is closely related to fretting fatigue. The partial slip contact problem is when both contact pressure and shear traction added on the contact pad (i.e. the cylinder), the introduction of shear traction will however decrease the contact surface and make part of the body start to slip. Recently, Professor David. A. Hills invented a method called dislocation method, and can be well used in 2D contact problem, with shear traction only applied in one direction, this project is to deeply investigate what is the contact behaviour when shear traction added in two directions and analytical dislocation method will be used in this project.

1.2 Mathematical Deduction

1.2.1 General contact Formulation

Let us start off with some basic equations for 3d contact problem, with x axis along the surface, y axis coming out of the half plane and z axis perpendicular to both x and y, in the half plane, the surface displacement are given by[2]

\[ v(x) = \frac{(\kappa + 1)\ln |x|}{4\mu\pi} + \frac{P(\kappa - 1)\text{sgn}(x)}{8\mu} \]  
\[ u(x) = -Q\frac{(\kappa - 1)\text{sgn}(x)}{8\mu} - \frac{P(\kappa - 1)\text{sgn}(x)}{4\pi\mu} \]  
\[ w(x) = -R\frac{\ln |x|}{\pi\mu} \]

where \( k = 3 - 4v \) and \( \mu \) is the modulus of rigidity which equals to \( \frac{E}{2(\nu+1)} \).

Then the rate of change of above displacements are:

\[ \frac{d}{dx}(u_1 - u_2) = -\frac{A}{4\pi} \int\frac{q(\xi)d\xi}{x-\xi} + Bp(x) \]
\[
\frac{d}{dx}(v_1 - v_2) = -\frac{A}{4\pi} \int p(\xi) \frac{d\xi}{x-\xi} - \frac{B}{4} q(x)
\]
(5)

\[
\frac{d}{dx}(w_1 - w_2) = -\frac{2}{\mu\pi} \int r(\xi) \frac{d\xi}{x-\xi}
\]
(6)

where \(A = \frac{2(k+1)}{\mu}\) and \(B = 0\) due to similar material.

1.2.2 Influence of both edge and screw dislocation

by using the inversion formula, solve equation (4) and (6), singular case:

\[
q(x) = \frac{1}{\pi \sqrt{a^2 - x^2}} \left[ C + \frac{E^*}{2} \int_{-a}^a \sqrt{a^2 - \xi^2} u(\xi) \frac{d\xi}{\xi - x} \right], \quad |x| < a
\]
(7)

\[
r(x) = \frac{1}{\pi \sqrt{a^2 - x^2}} \left[ C + \frac{\mu}{2} \int_{-a}^a \sqrt{a^2 - x^2} w(\xi) \frac{d\xi}{\xi - x} \right], \quad |x| < a
\]
(8)

where \(E^*\) is plane strain modulus, which equals to \(\frac{E}{1-\nu}\) and \(C\), the rigid body term is zero.

Note that the screw dislocation and the edge dislocations will happened simultaneously at all the non sticky region, but the shear traction \(q(x)\) and \(q(r)\) will be independent due to the orthogonality

Assuming \(b_x\) and \(b_z\) are burger vectors in edge and screw dislocations respectively:

\[
B_x(x) = \frac{db_x}{dx}, \quad B_z(x) = \frac{db_z}{dx}
\]
(9)

Then we substitute equation (9) into \(u'\) and \(v'\) in equations (7) and (8):

\[
q(x) = \frac{E^*}{2\pi \sqrt{a^2 - x^2}} \int_{-a}^{a} \frac{\sqrt{a^2 - x^2} B_x(\xi) d\xi}{x-\xi}, \quad |x| < a
\]
(10)

\[
q(x) = \frac{\mu}{2\pi \sqrt{a^2 - x^2}} \int_{-a}^{a} \frac{\sqrt{a^2 - x^2} B_z(\xi) d\xi}{x-\xi}, \quad |x| < a
\]
(11)
1.2.3 Correction of fully adhered solution

Consider first the solution for a fully adhered contact subject to both in-plane shear force $Q$ and an anti-plane shear force, $R$. For all half-plane contacts, independent of geometry, the shear traction distribution, for a contact of half-width $a$, due to subsequent application of shear forces, $Q$ and $R$, is given by:

$$r_{st} = \frac{R}{\pi \sqrt{a^2 - x^2}}, \quad q_{st} = \frac{Q}{\pi \sqrt{a^2 - x^2}}$$

(12)

The appropriate strain nucleus, needed to permit slip, and can be represented by both edge and screw dislocation, $b_x(\xi)$ and $b_z(\xi)$, located on the interface between two half-planes, glued together over the interval $[-a, a], (-a < \xi < a)$, but where the surfaces of the half-planes are free of traction outside this interval. Stick is to be preserved over a central region $-b < x < b$, so that dislocations must be distributed only over the two outer slip regions, and hence two corrective shear traction distribution in both direction, $q_c(x)$ and $r_c(x)$ is given by

$$q_c(x) = \frac{E^*}{2\pi \sqrt{a^2 - x^2}} \left[ \int_b^a \sqrt{\frac{a^2 - \xi^2}{\xi - x}} B_x(\xi) d\xi + \int_{-a}^b \sqrt{\frac{a^2 - \xi^2}{\xi - x}} B_x(\xi) d\xi \right]$$

(13)

$$r_c(x) = \frac{\mu}{2\pi \sqrt{a^2 - x^2}} \left[ \int_b^a \sqrt{\frac{a^2 - \xi^2}{\xi + x}} B_z(\xi) d\xi + \int_{-a}^b \sqrt{\frac{a^2 - \xi^2}{\xi + x}} B_z(\xi) d\xi \right]$$

(14)

The slip displacements themselves a constitute and even function of $x$ so that the dislocation density is odd, and therefore $B_z(x) = -B_z(-x)$, giving

$$q_c(x) = \frac{E^*}{\pi \sqrt{a^2 - x^2}} \int_b^a \sqrt{\frac{a^2 - \xi^2}{\xi + x}} B_x(\xi) \frac{1}{\xi - x} d\xi$$

(15)

$$r_c(x) = \frac{\mu}{\pi \sqrt{a^2 - x^2}} \int_b^a \sqrt{\frac{a^2 - \xi^2}{\xi + x}} B_z(\xi) \frac{1}{\xi - x} d\xi$$

(16)

According to Pythagoras theory and Coulomb friction law, applied only in the sliding region, giving

$$[r_{st}(x) + r_c(x)]^2 + [q_{st}(x) + q_c(x)]^2 = f^2 p(x), \quad b < |x| < a$$

(17)

Assuming $q_x(x)$ and $q_z(x)$ represent the total equivalent shear traction in $x$ and $z$ direction giving

$$q_x(x) = \begin{cases} q_{st}(x) + q_c(x) & , b < |x| < a \\ q_{st}(x) & , |x| < b \end{cases}$$

(18)

$$q_z(x) = \begin{cases} r_{st}(x) + r_c(x) & , b < |x| < a \\ r_{st} & , |x| < b \end{cases}$$

(19)

1.2.4 Orthogonality condition

According to the principle that shear traction is always opposite to the direction of displacement, giving

$$\tan \phi = \frac{q_z(x)}{q_x(x)} = \frac{b_z(x)}{b_x(x)} = \frac{\int B_z(x) dx}{\int B_x(x) dx} = \lambda(x)$$

(20)

also

$$\int_{-a}^a q_x(x) dx = Q$$

(21)

$$\int_{-a}^a q_z(x) dx = R$$

(22)
1.2.5 Application to Hertz’ problem

The contact pressure distribution in the Hertz’ problem, bounded solution, is given by

\[ p(x) = \frac{2P}{\pi a^2} \sqrt{a^2 - x^2} \]  \hspace{1cm} (23)

Using equation (12),(15)-(20) and (23), we show, in Appendix A and B

\[ B_x(t) = \frac{\sqrt{1+t}}{\pi^2(1-t^2)} \sqrt{\frac{2}{c-d}} \left[ \frac{2Pf\pi(c-d)(-t^2 + t + \frac{\pi}{2})}{E^*c\sqrt{\lambda^2 + 1}} - \frac{2Q\pi t}{E^*} \right] \]  \hspace{1cm} (24)

\[ B_z(t) = \frac{\sqrt{1+t}}{\pi^2(1-t^2)} \sqrt{\frac{2}{c-d}} \left[ \frac{2P\lambda f\pi(c-d)(-t^2 + t + \frac{\pi}{2})}{\mu c\sqrt{\lambda^2 + 1}} - \frac{2R\pi t}{\mu} \right] \]  \hspace{1cm} (25)

under the condition of \(-1 < |t| < 1\).

To verify the assumption made in appendix B, following requirement need to satisfy \(\lambda(x) = \lambda = \frac{R}{Q}\),

\[ \lambda = \frac{\int B_z(x)dx}{\int B_x(x)dx} = \frac{\int B_z(t)dt}{\int B_x(t)dt} \quad b < |x| < a \]  \hspace{1cm} (26)

1.3 Conclusion and Discussion

So far the math is very complicated especially in the verification process, it is meaningless to do further calculation if we cannot simplify the math, the integral result is horribly long. Therefore I think:

- The relationship between stick region \(b\) and \(Q,R,P\) give one more equation and probably can simplify the math, but I need further information to compute the relationship.
- We can directly use the numerical method to solve the equation.
- In terms of getting the precise answer, we could use something like Newton’s method, using the iterated approach, for any position \(b < |x| < a\), say \(x = t\) (just my guess)

1. Initially set \(\lambda_0 = \frac{R}{Q}\).
2. Calculate \(B_z(x)\) and \(B_x(x)\), then do the integral to get \(b_x(x)\) and \(b_z(x)\).
3. \(\lambda_1 = \frac{b_z(x)}{b_x(x)}\).
4. Repeating 3.1. to 3.3. to get \(\lambda_2\).

...  
5. Stop the iteration until \(\lambda_n = \lambda_{n+1} = \lambda(t)\).
2 3D contact problem

2.1 introduction

In many real situations, for example, the jet engine, can be modelled as a three-dimensional incomplete contact problem with finite length, as in figure 1. Turbine and compressor blades are often connected to the disk by a dovetail joint as depicted. As the gas turbine rotates, the dovetail geometry experiences centrifugal, $F_c$, and vibrational, $F_v$, loading. The centrifugal force brings the blade and disk into contact. The two established contact flanks are subject to normal and shear forces, here indicated by $P$ and $Q$, respectively[1]. Do not like 2D problem which was investigate in the first part, the 3D problem do not have a specific analytical result so far, due to the incomplete contact, the contact surface will change in response of variation of the applied load. The pressure distribution will change along the direction of model’s propagation, especially when it comes to the free edge there will be a sharp decrease of the contact pressure, thus there is a possibility that the contact pressure near the free edge is too low to hold the shear traction. The uncertainties introduced by increasing one dimension makes 3D contact problem the therefore the optimised case for the contact profile will be that can produce no pressure gradient and with a uniform rectangular shape contact pattern.

Figure 1: Illustration of a dovetail geometry with unequal radii subject to external loads

2.2 methodology

2.2.1 The Analytical Method and The Introduction to Finite Element Analysis

Hertzian contact in plane-strain condition can be characterized as following equations [2]:

$$A = \frac{k_1 + 1}{4\mu} + \frac{k_2 + 1}{4\mu}$$

$$a^2 = \frac{2AP}{\pi k}$$

$$p(x) = -\frac{2P}{\pi a} \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

where $k = 3 - 4v$ and $\mu$ is the modulus of rigidity which equals to $\frac{E}{2(1+v)}$; $a$ is the contact half-width and $P$ is the total applied load. $p(x)$ is the equation of contact pressure profile.

However, Above equations only summarized the Hertzian contact behaviour in two dimensional cases, the
three dimensional Hertzian contact is still not be completely probed analytically.

Finite Element Analysis (FEA) is an efficient method to simulate the mechanical behaviour of contact problems, so to enhance the understanding of three dimensional problem. the software used in this study is ABAQUS to help to stimulate all the models, and the variable we are interest in is Contact pressure at the contact surface (Cpress).

### 2.2.2 Two dimensional Finite Element (FE) model

two dimensional model was built to act as a reference to see the accuracy of the 3D models, as depicted in figure 2

![Figure 2: illustration of two dimensional model](image)

And the checking will also depicted in figure 3, and showing that except the edge convergence, the other parts are all fine.the difference in the edge part is mainly due to the mesh effect.

### 2.2.3 first stage 3 dimensional analysis

To have a deeper understanding about the problem, Nine 3D prototypes was built to evaluate the effect of change the contact surface and the contact pressure gradients.

![Figure 3: comparision between 3D, 2D and analytical solution](image)
The difficulties in the 3D model is that Abaqus can only provide 2d relation of contact pressure and the displacement, what we actually need in this case is to produce a 3d plot of the diagram.

Fortunately, the python code behind Abaqus is changeable, then the python script which describe the path of the entire surface was created and extract the data from python. To plot 3D, Matlab was used, simply imported data get from abaqus and import it into python and then figure was drawn depicted as figure 4 which shows a single case, and figure 5 which shows nine case combined together. Further observation will be made from those diagrams.

Figure 4: a typical example of 3d Matlab diagram

Figure 5: illustration of different stress have different effect on contact pressure and contact surface
2.2.4 Conclusion and Further Progression

So far the project went well and it will become my forth year project. However, no clear conclusion we can draw from observe above diagram, the reason is very simple: this is because the conclusion is mesh dependent, and to further progress we need to refine the mesh.

Increase the refinement of the model means introduce extra computational time. For example if we want to make the mesh 2 times finer, 7 times more time the model will spend to reach this object. For a relatively coarse mesh: 0.1mm, it already take 35 hours to run it. Thus improvement must made to reduce the computational time.

Robert Flicek used to write a method in his DPhil thesis about how to decrease degree of freedom from Abaqus [4], I will use the method to reduce the degree of freedom to save computational time.

Secondly, ways need to be created to generate a ‘plane’ contact surface to achieve parallel contact surface and no stress gradient. This, however, need further investigation.

References


Appendix A: Transformation to the integral equation

Combine equations (17), (18) and (19), giving

$$q_x^2(x) + q_z^2(x) = f^2 p^2(x)$$  \hspace{1cm} (30)

equation (20):

$$\lambda(x) = \frac{q_z(x)}{q_x(x)}$$  \hspace{1cm} (31)

then in the region $b < |x| < a$

$$[1 + \lambda^2(x)q_x^2(x) = f^2 p^2(x)$$  \hspace{1cm} (32)

$$\sqrt{1 + \lambda^2(x)q_x(x) = f p(x)}$$  \hspace{1cm} (33)

thus

$$\frac{E^* \sqrt{1 + \lambda^2(x)}}{\pi \sqrt{a^2 - x^2}} \int_b^a \frac{\sqrt{a^2 - \xi^2 B_x(\xi)\xi}}{\xi^2 - x^2} d\xi + \frac{Q \sqrt{1 + \lambda^2(x)}}{\pi \sqrt{a^2 - x^2}} = \frac{2fp}{\pi a^2} \sqrt{a^2 - x^2}$$  \hspace{1cm} (34)

$$\int_b^a \frac{\sqrt{a^2 - \xi^2 B_x(\xi)\xi}}{x^2 - \xi^2} d\xi = \frac{2fp(a^2 - x^2)}{a^2 E^* \sqrt{1 + \lambda^2}} - \frac{Q}{E^*} \quad b < |x| < a$$  \hspace{1cm} (35)

At the moment the kernel does not have a standard form, so we turn it into a Cauchy singular integral equation by dint of the transformations

$$a^2 = c ; b^2 = d ; x^2 = y ; \xi^2 = \eta$$  \hspace{1cm} (36)

so that

$$\frac{1}{2} \int_d^c \frac{\sqrt{c - \eta} B_x(\eta)}{y - \eta} d\eta = \frac{Q}{E^*} - \frac{2fp}{E^* \sqrt{\lambda^2(y) + 1}} \left(1 - \frac{y}{c}\right) \quad d < y < c$$  \hspace{1cm} (37)

The integral is normalised over the interval $[-1, 1]$ so we can use standard inversion procedures using the substitutions

$$2\eta = (c - d)s + (c + d); \quad 2y = (c - d)t + (c + d)$$  \hspace{1cm} (38)

$$\int_{-1}^1 \frac{\sqrt{1 - s} B_x(s)}{t - s} ds = \sqrt{\frac{2}{c - d}} \left[\frac{2Q}{E^*} - \frac{2fp}{E^* \sqrt{\lambda^2(t) + 1}} \left(1 - \frac{d}{c}\right)(1 - t)\right], -1 < t < 1$$  \hspace{1cm} (39)

Appendix B: Solve the integral equation

If $\lambda(x)$ is a function of $t$, then the integral equation is impossible to be solved analytically, thus we made following assumption:

$\lambda(x)$ is a constant value and

$$\lambda(x) = \lambda = \frac{R}{Q}$$  \hspace{1cm} (40)

And the inversion formula, itself gives

$$B_x(t) = \frac{\sqrt{1 + t}}{\pi^2 (1 - t^2)} \sqrt{\frac{2}{c - d} \left[\frac{2Q\pi(c - d)(-t^2 + t + \frac{c}{d})}{E^* c\sqrt{\lambda^2 + 1}} - \frac{2Q\pi t}{E^*}\right]}$$  \hspace{1cm} (41)
Similarly

\[ 1 + \frac{1}{\lambda^2} q_2^2(x) = f^2 p^2(x) \]  

(42)

The inversion formula gives:

\[
B_z(t) = \frac{\sqrt{1 + t}}{\pi^2 (1 - t^2)} \sqrt{\frac{2}{c - d}} \left[ \frac{2 P \lambda f \pi (c - d) (-t^2 + t + \frac{c}{2})}{\mu c \sqrt{\lambda^2 + 1}} - \frac{2 R \pi t}{\mu} \right]
\]  

(43)

in both cases

\[
c = a^2 \quad d = b^2
\]  

(44)

\[
t = \frac{2x^2 - a^2 - b^2}{a^2 - b^2}
\]  

(45)